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NOTE

A MIN-MAX RELATION FOR MONOTONE PATH SYSTEMS IN SIMPLE REGIONS

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A monotone path system (MPS) is a finite set of pairwise disjoint paths (polygonal arcs) in the plane such that every horizontal line intersects each of the paths in at most one point. We consider a simple polygon in the xy-plane which bounds the simple polygonal (closed) region D. Let T and B be two finite, disjoint, equicardinal sets of points of D. We give a min-max relation for the maximum number of points of T and B which can be joined by a MPS in D, and a polytime algorithm for finding such a MPS.

All of this paper takes place in the xy-plane. We will freely use the words up, down, left, right, horizontal, highest, etc.

We will consider a simple polygon Δ in the xy-plane. Δ and its interior is called a polygonal region, which we will denote by D.

A *monotone path* π is a polygonal arc of positive length such that every horizontal line intersects π in at most one point. A monotone path *in* D is a monotone path whose interior is contained in the interior of D. A *monotone path system* (MPS) Π (in D) is a finite set of pairwise disjoint monotone paths (in D). The sets of top points and bottom points of paths in Π are denoted by $T(\Pi)$ and $B(\Pi)$, respectively. We say that Π pairs $T(\Pi)$ with $B(\Pi)$.

Problem 1. Given two sets, T and B, of points of polygonal region D, find a MPS Π in D which joins as many points of T as possible to points of

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B; that is, find Π such that $T(\Pi)\subseteq T$, $B(\Pi)\subseteq B$, and $T(\Pi)$ is as large as possible.

Throughout this paper we assume that T and B are two finite, disjoint, equicardinal sets of points of D such that no point of T is locally lowest in D and no point of B is locally highest in D. Region D with points of $T\cup B$ inserted will be denoted by $D^{T,B}$.

A *horizontal chord* C of D is a horizontal line segment of positive length whose interior is interior to D and whose end points are on Δ. Horizontal chord C partitions D into three disjoint parts: C; the upper component, $U(C)$, of $D-C$, which is the component of $D-C$ for which C is a set of locally lowest points of $U(C)\cup C$; and the lower component, $L(C)$. Define

$$\begin{aligned} d(C) &:= |U(C)\cap T| - |(U(C)\cup C)\cap B| \\ &= |L(C)\cap B| - |(L(C)\cup C)\cap T|. \end{aligned}$$

A *peak (valley)* is a unique locally highest (lowest) point of D. A *reverse peak (reverse valley)* p is a vertex of Δ, which is a unique locally highest (lowest) point of Δ, but not a locally highest (lowest) point of D. A *problem point* is a reverse peak in T or a reverse valley in B. (See Figure 1.)

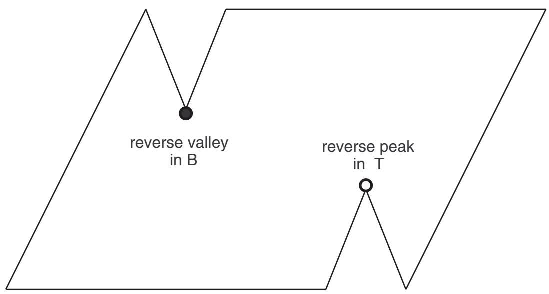

Fig. 1. Problem points.

A *distinguished horizontal chord* (DHC) of $D^{T,B}$ is a horizontal chord of D containing at least one point which is either in $T\cup B$ or is a vertex of Δ.

In [1], the following is proved.

Theorem 1. Where T and B contain no problem points, there is a monotone path system Π in D which pairs T with B if and only if $d(C)\geq 0$, for every distinguished horizontal chord C of $D^{T,B}$.

A *distinguished segment*, K, of $D^{T,B}$ is a horizontal line segment of D, whose endpoints are on $\varDelta$, whose intersection with $\varDelta$ is a set of isolated points, and which contains at least one DHC. Note that K consists of a sequence of adjacent DHCs. Define

$\bar{d}(K) := \sum\{d(C) : C$ is a horizontal chord on $K\}+$ the number of problem points on K.

In [1], the following is proved.

Theorem 2. *There is a monotone path system $\varPi$ in D which pairs T with B if and only if $\bar{d}(K) \geq 0$, for every distinguished segment K of $D^{T,B}$.*

The proofs of Theorems 1 and 2 provide polytime algorithms to find either an MPS or a bad chord or segment (i.e. in the case of no problem points, a chord C with $d(C) < 0$; otherwise a segment K with $\bar{d}(K) < 0$).

Consider a bipartite graph $G = ((T, B), E)$ with node-sets T and B and an edge between $t \varepsilon T$ and $b \varepsilon B$ if there is a monotone path π in D with top point t and bottom point b. A matching is a set of edges which meets each node at most once. For a matching M in G, let $T(M)$ be the set of points of T met by M, and define $B(M)$ similarly.

Lemma 1. *M is a matching in G if and only if there is an MPS $\varPi$ in D with $T(\varPi) = T(M)$ and $B(\varPi) = B(M)$.*

Proof. $\Leftarrow$ is obvious.

$\Rightarrow$: Let M be a matching in G. Say $T(M) = \{t_1, \cdots, t_m\}$, and $B(M) = \{b_1, \cdots, b_m\}$, and $\{t_i, b_i\} \varepsilon M$, $1 \leq i \leq m$. By Theorem 2, we must prove that for every distinguished segment K in $D^{T(M),B(M)}$, $\bar{d}(K) \geq 0$.

For a chord C of $D^{T(M),B(M)}$,

$$\begin{aligned} d(C, D^{T(M),B(M)}) &:= |T(M) \cap U(C)| - |(B(M) \cap (U(C) \cup C)| \\ &= \sum_{i=1}^{m} (|\{t_i\} \cap U(C)| - |\{b_i\} \cap (U(C) \cup C)|) \\ &= \sum_{i=1}^{m} d(C, D^{\{t_i\},\{b_i\}}). \end{aligned}$$

For a distinguished segment K of $D^{T(M),B(M)}$,

$$\begin{aligned} &\bar{d}\left(K, D^{T(M),B(M)}\right) \\ &\quad := \sum \left\{ d\left(C, D^{T(M),B(M)}\right) : C \text{ is a horizontal chord on } K \right\} \\ &\qquad + \text{the number of problem points of } T(M) \cup B(M) \text{ on } K \\ &\quad = \sum_{i=1}^{m} \left[\sum \left\{ d\left(C, D^{\{t_i\},\{b_i\}}\right) : C \ \text{ is a horizontal chord on } K \right\} \right. \end{aligned}$$

$$+ \text{the number of problem points of } \{t_i, b_i\} \text{ on } K\Big]$$

$$= \sum_{i=1}^{m} \bar{d}\Big(K, D^{\{t_i\},\{b_i\}}\Big).$$

Since $\{t_i, b_i\} \in M$, there is a monotone path in D from t_i to b_i, thus $\bar{d}\Big(K, D^{\{t_i\},\{b_i\}}\Big) \geq 0$. Thus $\bar{d}\Big(K, D^{T(M),B(M)}\Big) \geq 0$. Thus there is a MPS Π in D with $T(\Pi) = T(M)$ and $B(\Pi) = B(M)$. ∎

Here is how we solve the problem of finding an MPS in D which pairs as many points of T and B as possible. Create the bipartite graph $G = ((T, B), E)$ described above. Find a largest matching M in G. Apply the algorithm of [1] to $D^{T(M),B(M)}$ to find a MPS Π in D with $T(\Pi) = T(M)$ and $B(\Pi) = B(M)$. Then Π joins as many points of T and B as possible.

To create G, we must determine, for each $t \in T$ and $b \in B$, if there is a monotone path in D from t to b. This can be done by applying the algorithm of [1]. (We would find the "trapezoidization" or "canonical dissection" of D and then for each $t \in T$ and $b \in B$, insert t and b, and apply the rest of the algorithm.) It follows from the analysis of the algorithm in [1] that this can be done in $O(m^2 n \log n)$ time where $m = |T|$ and n is the number of vertices of D. We then find a largest matching in G, which can be done in $O(m^{5/2})$ time [5]. Then apply the algorithm of [1] to $D^{T(M),B(M)}$, which can be done in $0\Big((n + |T(M)|) \max \Big\{|T(M)|, \log(n + |T(M)|)\Big\}\Big)$ time which is $\leq 0\Big((n+m) \max \{m, \log(n+m)\}\Big)$ time. Thus the overall complexity of the algorithm is:

$$O\Big(\max\Big\{m^2 n \log n, m^{5/2}, (n+m) \max\{m, \log(n+m)\}\Big\}\Big)$$
$$\leq \begin{cases} O(m^3 \log n) & \text{if } m \geq n \\ O(m^2 n \log n) & \text{if } m \leq n. \end{cases}$$

The König–Hall Theorem [6,4] says that the maximum size of a matching in a bipartite graph H equals the minimum number of nodes which meet all the edges of H. Interpreting this for our bipartite graph G, we obtain:

Theorem 3. *The maximum size of a MPS Π in D with $T(\Pi) \subseteq T$ and $B(\Pi) \subseteq B$ equals the minimum number of points of $T \cup B$ which meet every monotone path in D from T to B.*
Equivalently, for every positive integer k, either there is a MPS Π in D with $|\Pi| = k$, $T(\Pi) \subseteq T$ and $B(\Pi) \subseteq B$, or there is a set $S \subseteq T \cup B$, $|S| < k$, which meets every monotone path in D from T to B. Not both.

Theorem 3 is a good characterization in the sense of Edmonds [2]. To see that $S \subseteq T \cup B$ meets every monotone path from T to B, we can display for each $t \in T - S$ and $b \in B - S$, a bad chord or segment in $D^{\{t\},\{b\}}$. (The algorithm of [1] finds this.)

We now look at the following more general problem:

Problem 2. Given integral weights w_v on the ponts $v \in T \cup B$, find a MPS in D which pairs a maximum weight set of points of T and B.

This can be done by applying the algorithm described for Problem 1 except give edge $e = \{t, b\} \in E$ weight $w_e = w_t + w_b$, and then find a maximum weight matching in G. A maximum weight matching in a bipartite graph with k vertices in each part can be found in $O(k^3)$ arithmetic operations [7]. Thus the overall number of arithmetic operations of this algorithm for Problem 2 is:

$$O\Big(\max \Big\{ m^2 n \log\, n, m^3\,, (n+m)\, \max\{m, \log(n+m)\} \Big\} \Big)$$
$$\leq \begin{cases} O(m^3\, \log n) & \text{if } m \geq n \\ O(m^2 n\, \log n) & \text{if } m \leq n \end{cases}$$

For a bipartite graph $H = (V, E)$ with integral weights w_e on the edges $e \in E$, the König-Egerváry Theorem [6,3] says:

$$\text{maximum} \left\{ \sum_{e \in M} w_e : \text{is a matching in H} \right\}$$
$$= \text{minimum} \left\{ \begin{array}{l} \displaystyle\sum_{v \in V} y_v : \forall e = \{u, v\} \in E, y_u + y_v \geq w_e \\ \qquad\qquad \forall v \in V, y_v \geq 0, \quad y_v \text{ integer} \end{array} \right\}$$

For G with the weights as described above, the König–Egerváry Theorem becomes the following:

Theorem 4: The maximum weight of a set of points of $T \cup B$ which can be paired by a MPS in D equals the minimum sum $\sum_{v \in T \cup B} y_v$ of non-negative integral weights y_v of $v \in T \cup B$ such that if there is a monotone path from t to b in D, then $y_t + y_b \geq w_t + w_b$.

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